CHAPTER 3

Elastic Body Mechanics

ELASTIC BODIES

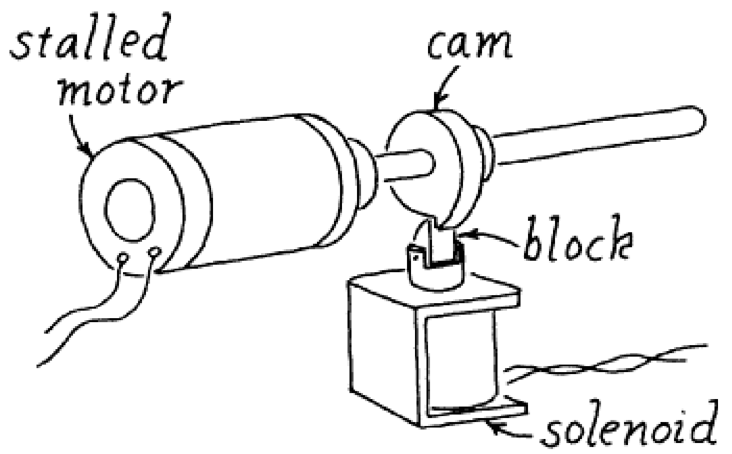
Rigid Body Mechanics Not Sufficient

Thus far, only conventional, or rigid body mechanics has been considered; the type learned daily by undergraduates. But it is also the kind that fails to solve real-life problems which occur in dealing with many types of intermittent motion mechanisms, or any high-speed mechanism that involves a lot of clearance, backlash, and/or sudden changes in velocity. To see why rigid body mechanics is not always sufficient, consider the action of a simple escapement mechanism. Figure 3–1 shows a shaft that is continuously urged to rotate by a stalled electrical motor. A cam on the shaft contacts a block which normally prevents rotation of the shaft. When the block is momentarily pulled out of the way by a solenoid the shaft will turn. After one turn the cam strikes the block and output motion ceases. (The solenoid, of course, has to be de-energized soon enough to let the block return to the interference position before the cam completes its first revolution, if only one turn is desired.) Not a very elegant mechanism, but it does produce intermittent motion, and it is very simple.

The torque-time curve for the control cam of this escapement is shown in Fig. 3–2. When the block is withdrawn, the motor produces a torque on the cam and shaft, and these members then will move (with steadily increasing velocity). When the cam strikes the re-positioned block, however, it must stop instantly to conform to the fact that the block has zero velocity. (Or the block must suddenly acquire the velocity of the cam, which the designer presumably does not want.)

But as was seen in the previous chapter, an instantaneous stop would mean an infinite negative acceleration, hence, an infinite torque. These things, however, are just not possible. What actually does happen, is that each of the colliding bodies, the cam and the block, try to alter their respective velocities to conform to that of the other. They ultimately succeed. The length of time this "getting together" takes, and the forces that are generated during this process depend on such factors as the stiffness, inertia, and impact velocities of both bodies and, to a lesser extent, upon the frictional and drive torques which are acting on the bodies during the collision. It seems a little strange to have to say "to a lesser extent" about something as tangible as a drive force, but most practical drive forces are so small in comparison to actual impact forces that they have little influence on impact duration times or peak forces.

The impact forces that are generated in collisions of this type get to be extremely high, even in small, lightweight instruments driven by what appear to be relatively weak drivers. In most machine applications, in fact, the forces that are generated as the bodies struggle to conform to each other's desires are so high as to deform the bodies. Once this happens, however, we are dealing with elastic body mechanics instead of rigid body mechanics.



Drawing Courtesy of MACHINE DESIGN Magazine; Nov. 21, 1968

Fig. 3-1. Simple electro-mechanical intermittent motion mechanism.

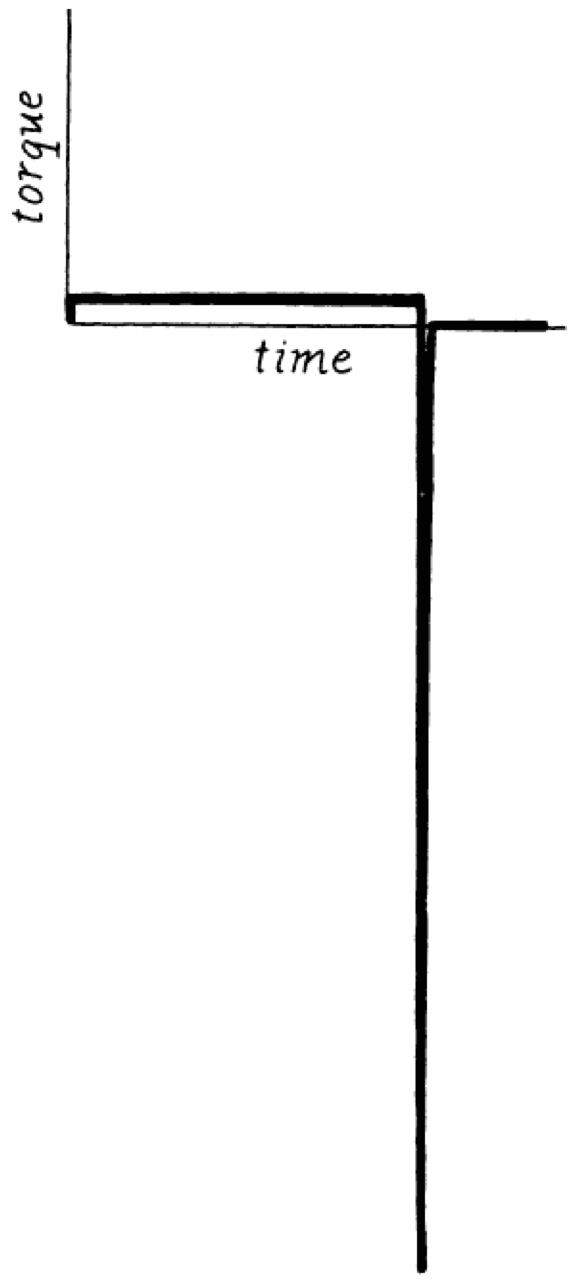
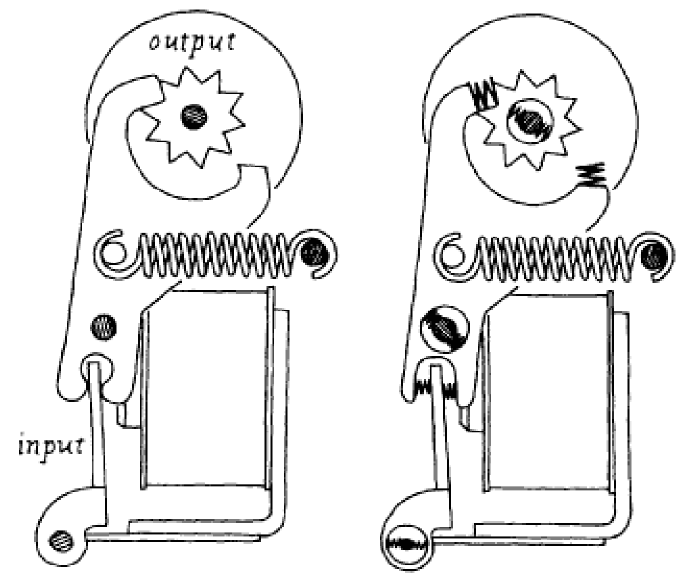


Fig. 3-2. Torque-versus-time curve for the cam of Fig. 3-1.

The motions of machine bodies and the forces which they exert on each other, as well as the internal stresses within the bodies, are all modified by the fact that the bodies are now deforming as they operate. Each link in a mechanism train behaves like a spring; storing, releasing, and dissipating energy. Designers have found from experience that if they try to treat the bodies as pure springs for computational purposes, they will get into all sorts of complications and confusions. In most design situations it is sufficient, instead, to find a substitute "model" for the body which is easier to understand than a pure spring and which is easier to handle from a theoretical standpoint. One of the most useful of such models is called the *quasi-static model.*



Drawing (Left) courtesy of MACHINE DESIGN Magazine; Dec. 23, 1965; p. 121ff

Fig. 3-3. (*Left*) Inverse escapement. (*Right*) A quasi-static model of the same mechanism.

The Quasi-Static Model

Machine members can be thought of as rigid bodies connected by small springs, as shown in Fig. 3-3. The masses in this model are assumed to be perfectly rigid and to obey all of the laws of rigid body mechanics. All elasticity of the machine members is assumed to lie in small springs that connect the rigid masses together. Mathematicians call this a quasi-static model of elastic bodies and find it very useful in approximating the performance of elastic bodies in actual design situations.

In the official quasi-static model the springs are assumed to be linear; that is, if the forces acting on the springs are doubled, their deflection is doubled. They are also assumed to be "perfect" springs, which means that no energy is lost within the spring as it deforms. This latter is necessary in order to simplify the equations used to study such bodies.

However, where an approximate understanding only, is sufficient, the fact that some energy is lost during every collision or interaction between two members of a machine can be assumed. As in the model in Fig. 3–3, therefore, each of the various machine links is taken to be rubbing on a frictional surface as it moves. These fictitious surfaces do not confine the motion of the linkage; they merely remind us that energy is being lost within or between these various elements of the machine as they perform their jobs.

For completeness, it should be added that engineers have built other elastic body "models" found to be more accurate for certain situations than the quasi-static model. One of these, for example, takes account of the fact that permanent deformation of one or both bodies can occur during certain types of impact. It is desirable of course, to avoid permanent deformation of machine members and thus, I think that particular model can safely be ignored.

Another model assumes that the springs connecting the bodies are not normal springs, but have nonlinear characteristics; the force required to deflect the spring is not just a linear function of the deflection. This situation is clearly beyond the scope of this text, and besides, the point is to get approximate understanding and not exact answers, in any case.

Using the Quasi-Static Model

First, let us see how to use the quasi-static model to estimate the torque exerted on the cam when it strikes the block in Fig. 3–1. Assume that the motor exerts a net drive torque of 0.5 ft-lbs on the cam, and that it takes the cam one second to complete its first revolution. Assume also, that a limit switch, set next to the block, turns off the motor the instant the cam strikes the block to avoid complicating the calculation with drive forces. As will be seen later, this is not a bad assumption since the impact forces will be much higher than the drive forces.

Assume next that the cam is brought to rest by the block in 10 milliseconds. This is not a bad assumption in a typical machine design impact; but later on, shorter and longer times will also be considered. Finally, assume that the inertia of the cam, cam shaft, motor rotor, and everything else being brought to a stop by the block is 0.5 slug-ft^2.

The positive area under the torque–time curve of Fig. 3–2 is:

$$\tau_{\text{drive}} \times t_{\text{drive}} = 0.5 \text{ ft-lbs} \times 1 \text{ sec}$$

or 0.5 ft-lb-sec. In order for the block to stop the cam it must exert sufficient torque for sufficient time on the cam to produce a negative torque–time area equal to the positive torque–time area as was seen when studying the finger-pushing-a-block case in Chapter 1.

A stopping time of 10 milliseconds has been assumed, then as a first approximation, merely divide the positive area of the torque–time curve by the stopping time to get the average stopping torque.

$$\text{Positive area} = \text{Negative area}$$

or:

$$\text{Drive torque} \times \text{Drive time} = \text{Stopping torque} \times \text{Stopping time}$$

or:

$$\text{Stopping torque} = \frac{\tau_{\text{drive}} \times t_{\text{drive}}}{t_{\text{stop}}} = \frac{0.5 \text{ ft-lbs-sec}}{0.010 \text{ sec}}$$

$$\text{Stopping torque} = 50 \text{ ft-lbs (average)}$$

This is not a very good approximation, however, because it assumes that the stopping force is constant during the entire stop. The quasi-static model suggests that during the collision the surfaces of the bodies act like springs. The fact is that it takes a very long, soft spring to produce a "flat rate" that is independent of deflection. As most springs deflect, the force required for further deflection increases. And, of course, most machine members are very stiff; they have very high spring rates and resist deflection vigorously. A tiny deformation (fractions of a thousandth of an inch) requires extremely high forces (thousands of pounds) in actual designs involving "rigid" bodies.

If it is assumed that the actual deformation of the cam and block of Fig. 3–1 involves stiff, linear-spring-rate members as suggested by the quasi-static model, then the build-up of stopping force as a function of time will be sinusoidal as in Fig. 3–4, rather than a "square wave" as in the initial assumption when constant torque was assumed during the entire stop.

Note that the torque–time curve is sinusoidal, not triangular. Most designers think of springs as producing triangular curves—and, indeed, they do produce triangular torque (or force) versus *displacement* curves. But the torque–time curve for a linear spring is always sinusoidal.

The block must still produce the same average 50 ft-lbs of torque if the cam is to be brought to rest in ten milliseconds, however. If the relationship between the average and peak values for one-quarter cycle of the sine wave is known, a factor of proportion can be found that will allow an estimate of the actual peak force that is a little more accurate than was found at first. This factor of proportionality turns out to be about 0.63. (This can be determined graphically by drawing one quarter of a sine wave and computing its average height above the horizontal axis.) The peak torque during the impact now appears to be:

$$\text{Max. torque} = \frac{\text{Average torque}}{0.63} = \frac{50\text{ ft-lbs}}{0.63}$$

$$\text{Max. torque} = 79\text{ ft-lbs}$$

or approximately 80 ft-lbs. And all of this has been generated by a $\frac{1}{2}$ ft-lb driver acting for only one second on a $\frac{1}{2}$ slug-ft^2 cam. Now it can be seen why it was quite safe to ignore drive torque when calculating stopping torque.

The cam has now been brought to rest in the process. However, the small spring assumed to exist between the colliding surfaces has been "loaded." Since it is assumed that the bodies are elastic, the motion stops only for an instant. The spring will now push back on the cam as it unloads, completing the sinusoidal curve started during the first half of the collision (Fig. 3-5). In other words, the cam bounces off the block. If the bodies were fully elastic, the cam would rebound with a velocity equal (but

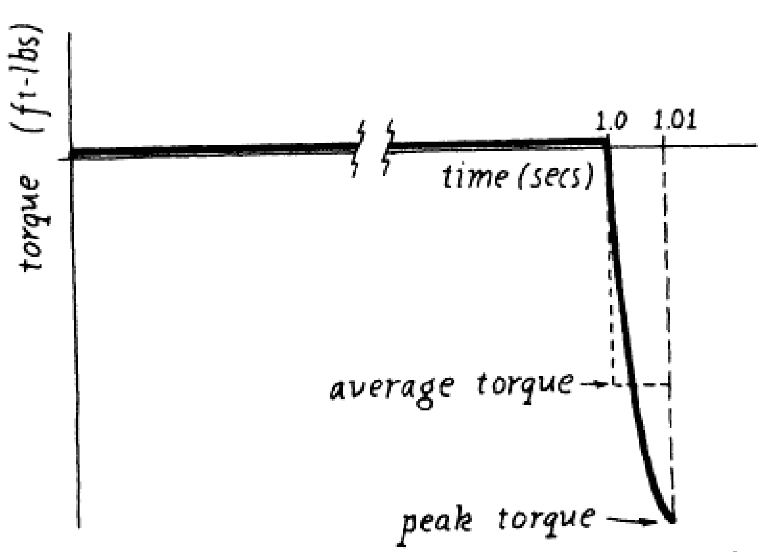


Fig. 3-4. Detail: average torque versus peak torque for the mechanism shown in Fig. 3-1.

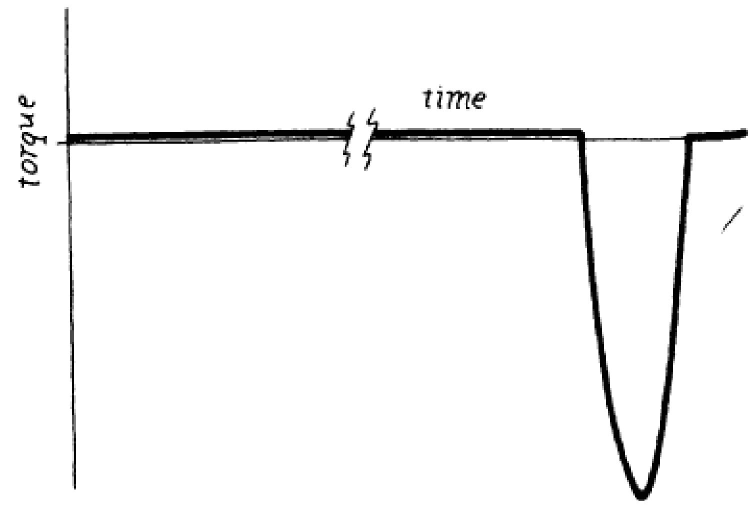


Fig. 3-5. Detail: complete sinusoidal torque-versus-time curve for the mechanism of Fig. 3-1.

opposite) to the velocity it had upon colliding with the block. In actual machine design situations, however, the rebound velocity will be only a small percentage (perhaps 10 or 20 percent) of the initial velocity; the rest of the impact energy will be lost in internal and external friction during the collision (and the final portion of the curve in Fig. 3-5 would be altered to reflect this).

We seem to be getting closer to an "exact solution" for the maximum stopping force in this case; but it must be remembered that we started with the arbitrary assumption that the cam was brought to rest by the block ten milliseconds after they touched. If the actual time was 100 milliseconds (which is probably longer than the impact times involved in most machine body collisions) the peak torque would be only 8 ft-lbs. If, on the other hand, the cam were brought to rest in one millisecond (which is in the ballpark for some machine collisions), the peak torque in this same situation would be 800 ft-lbs: Thus, the initial assumption of an impact-duration time has a large bearing on the "answer" obtained. But, hopefully, the graphical process has provided some insight, at least, into what is important and what is not.

Another Point of View

Sometimes we do not know what forces or torques produced the motion of impacting bodies, nor for how long such forces were acting. All we have are experimental data on the relative velocities of the bodies (in the previous example this would be the velocity of the cam) prior to impact. Since we know that impacting bodies can be treated as quasi-static