CHAPTER II.

POSITION, VELOCITY, AND ACCELERATION.

11. Velocity.—While Kinematics in its general sense comprises all kinds of problems dealing with pure motion, the number of such problems falling within the province of the Kinematics of Machines is somewhat limited. We shall consider in this chapter some elementary notions concerning velocity which are applicable to the purposes of the Kinematics of Machines. Methods of studying the position and motion of a point or rigid body from a geometrical point of view have already been indicated; it now remains to investigate not only the amount by which such position is changed during motion, but the rate of such change of position. Going a step farther still, it may be asked, does such velocity increase, diminish, or change in any way as time goes on, and if so, at what rate?

The rate of change of position of a point or body is called its *velocity*. A body, as we have seen, may change its position by a motion of translation, or by one of rotation. Hence we distinguish between linear and angular velocity. The former is measured by the space passed over in unit of time, and is usually expressed in feet per second, although other units, such as miles per hour or knots, are adopted in special cases. The latter is measured by the angle described in unit of time, the natural unit being therefore one radian per second. Engineers, however, commonly measure angular velocity in revolutions per minute. Either kind of velocity may be uniform or variable.

It is important to note that the term velocity involves the ideas of both speed, direction, and sense. In other words, a velocity is a vector quantity, and, like other vector quantities, may be represented by a straight line of definite length, this length being proportional to the speed, or magnitude of the velocity, measured in feet per second, radians per second, or whatever units are to be employed.

In the case of linear velocity the direction of the vector or straight line representing the velocity on the diagram is taken to represent the *direction* of the motion. Thus, for example, we might draw upon a map a line running east and west, and 2 inches in length, and take this line as representing a linear velocity of 2 miles per hour, or 2 feet per second, either from east to west, or from west to east. The *sense* of the motion may be either from east to west, or from west to east. In order to indicate the sense, we place upon the line a small arrow-head so as to show the point towards which the body is moving (see Fig. 12).

In the case of angular velocity the direction of the vector on the diagram would be taken to represent the direction in space of the axis about which the spin or rotation is taking place, and a line similar to that mentioned above would mean a spin of two radians per second, or two revolutions per minute, according to the scale, about an axis lying east and west. This rotation may be either right-handed or left-handed, and it is therefore customary to indicate the sense by placing the arrow-head in such a fashion that the spin will appear to be right-handed, or clockwise, when looking along the axis and following the arrow-head.

It is plain that in this manner a velocity, whether linear or angular, may be completely represented by a vector, having magnitude, direction, and sense.

12. Uniform Velocity.—A body having uniform velocity (whether angular or linear) performs equal changes of position in equal times. If the body has a uniform linear velocity v, it describes a distance vt in time t, where t is any number of units of time. Calling s the space described, we have therefore $s = vt$. Similarly, if the uniform velocity is angular and is denoted by ω, any line on the body in a plane perpendicular to the axis of rotation describes ω radians in each second and therefore ωt radians in t seconds. Hence, calling θ the angle described in t seconds, we have

$$\theta = \omega t.$$

If a point, at distance r from the centre about which it moves in a circular path, has a linear velocity v, its angular velocity is measured by the angle subtended at the centre by the path it describes in one second. Hence

$$\omega = \frac{v}{r} \quad \text{or} \quad v = \omega r.$$

13. Variable Velocity. — In general a moving body varies its speed as well as its direction of motion. It is easy by observing the time taken to travel over a known distance, for example in a train, to calculate the *average speed* of the train during the interval considered. This does not tell us, however, the *actual speed of the train at any instant* during the interval of time, which may be quite different from the average speed.

The velocity at any instant, or instantaneous velocity, is measured by the space (or angle, as the case may be) which would have been described in a unit of time if the motion had continued uniformly, during that interval, at the same rate as at the instant considered. The word instant is here used to mean an indefinitely small interval of time.

We are not able to measure the distance (or angle) described during an indefinitely small interval of time, and therefore have to obtain the value of the instantaneous velocity of a body in another manner.

This will be best understood by a numerical example. Suppose that a man in a street-car at 12 o'clock finds that in 10 seconds the car traverses a distance of 200 feet. This gives 20 feet per second as the average speed during the 10 seconds after 12 o'clock. Suppose that other observations taken during the first $1\frac{1}{5}$, 2, and 4 seconds showed that during these times the car travelled 30, 48, and 100 feet, corresponding to average speeds of 25, 24, and 21.75 feet per second. It is evident that the speed must really have been continually diminishing, and that the shorter the time during which the observation was made, the more nearly do we

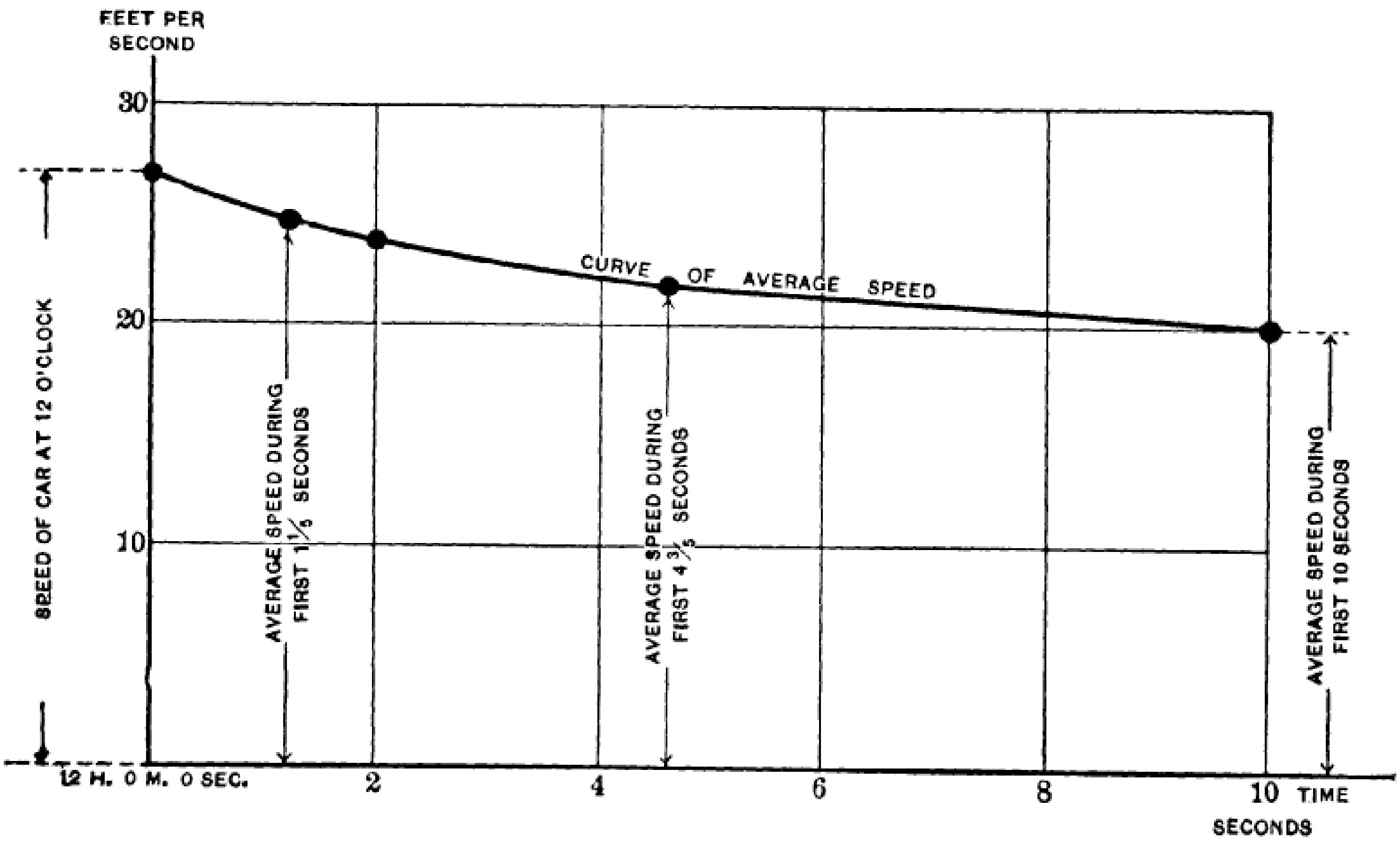

FIG. 10.

obtain the speed at which the car must have been travelling at 12 o'clock. To arrive at this more exactly, since we cannot measure the distance passed over in an infinitely small interval of time, we plot a curve from our observations, as in Fig. 10, and see that the speed at 12 o'clock must have been 27 feet per second. In mathematical language, if Δs be the distance traversed in a small interval of time Δt, the average velocity during that small interval is $\frac{\Delta s}{\Delta t}$, while the velocity at the instant beginning the interval is measured by diminishing Δt indefinitely, and finding the limiting value of $\frac{\Delta s}{\Delta t}$, or, in the language of the calculus $\frac{ds}{dt}$. Thus

$$v=\frac{ds}{dt}.$$

The same reasoning, of course, applies in the case of angular velocity, where we should write

$$\omega=\frac{d\theta}{dt}.$$

Compare these with the corresponding expressions in the case of uniform velocity.

14. Uniform Acceleration.—A body moving with uniform acceleration changes its velocity by equal amounts in equal times. Thus suppose that in time t the velocity changes from v_1 to v_2; we have, if a is the acceleration,

$$a=\frac{v_2-v_1}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, the average velocity during the time t is $\frac{v_2+v_1}{2}$, the arithmetical mean between the initial and final velocities; hence if s be the space described,

$$s=\frac{v_2+v_1}{2}t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From these two expressions we find

$$s=\frac{v_2{}^2-v_1{}^2}{2a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But $v_2 = v_1 + at$. Substituting in (3), we get

$$s = v_1 t + \frac{a}{2} t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the case of angular velocity precisely similar relations hold, so that, calling α the uniform angular acceleration, ω the angular velocity at the beginning of the time t, and θ the angle described, we have, instead of (4),

$$\theta = \omega_1 t + \frac{\alpha}{2} t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

To express the velocity in terms of distance (or angle) and initial velocity we shall have instead of (3)

$$\omega_2{}^2 = \omega_1{}^2 + 2\alpha\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

while the expression connecting velocity, acceleration, and time is

$$\omega_2 = \omega_1 + \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

As an example of the use of these expressions, suppose a wheel is revolving thirty times per second and comes to rest in 12 seconds. How many revolutions will it make in coming to rest if uniformly retarded?

We have $\omega_2 = \omega_1 + \alpha t$: hence

$$12\alpha + 30 \times 2\pi = 0,$$

and $\alpha = -\frac{60\pi}{12} = -15.71$ radians per second per second.

Again, $\omega_2{}^2 = \omega_1{}^2 + 2\alpha\theta$; hence

$$(60\pi)^2 - 10\pi\theta = 0,$$

and $\theta = \frac{(60\pi)^2}{10\pi} = 360\pi$ radians.

Hence the wheel comes to rest in 180 revolutions.