

CHAPTER VII.

DYNAMICS OF MECHANISM.

§ 23. LINEAR AND ANGULAR VELOCITY.

WE have now completed our examination of the nature of mechanisms, as well as of a series of the principal kinematic problems connected with them. So far we have been able to work by methods which are in reality purely geometrical, and have not found it necessary to introduce questions of force or mass at all except to show that they might—for that part of our work—be put upon one side. We now leave the Kinematics for the Dynamics of Mechanism, and come to problems which involve directly questions connected with forces and the balancing of forces. Before we are in a position to deal with these problems it will be necessary to give some further attention to the meanings and relations of the ideas which are involved in them, velocity, acceleration, force, mass, and so on.

We have hitherto looked at motion merely as change of position, and in reference to velocity we have only noticed the relative (and not the actual) velocities of different points in the same body or mechanism at the same time. We have now to deal with problems which involve the determination of actual velocities and the velocities of the same

point at different instants, problems which cannot be understood or solved without continual reference to the forces causing motion or change of motion.

We have distinguished (§ 15) between linear velocity and angular velocity. The former has to do with the distance, measured in ordinary length units, as feet, traversed by a point in a given time, while the latter is measured by the angle swept through by the radius of the point in a given time. Although something has already (in § 15) been said about the relation between linear and angular velocities, it is necessary to look at it here somewhat more in detail.

We have already seen (§ 7) that every body which has plane motion must be rotating about a point¹ in the plane. If this point be infinitely distant (like the virtual centre of a sliding pair), all the points in the body are moving at every instant in parallel straight lines and with the same velocity. In this motion of translation (which is thus merely a special case of motion of rotation) the velocity of the body is fully known if that of any one of its points be known. *The rate of motion of any point in its given direction of motion is the linear velocity of the point*, and in this case, to which we shall in the first instance confine ourselves, the linear velocity of any point is also the linear velocity of the whole body.

When we say that a body has a (linear) velocity 5, we mean that it moves at a rate which, if continued unchanged for a unit of time, would carry it through five units of distance in the given direction. The units of time and distance are commonly seconds and feet respectively, so that in the case supposed we should mean that the rate of motion

¹ The body, of course, rotates about an *axis*, but we have seen that the point which is the intersection of this axis with any plane parallel to the plane of motion, may be taken instead of the axis if we take instead of the body its section by or projection on the plane. See § 9.

of the point was such as would carry it through five feet in a second if it continued unchanged for that period.

But from the mere statement that a body has a linear velocity 5 *at a given instant*, we cannot infer that it possesses that velocity during any length of time, that, for instance, it will actually move five feet in a second, or still less that it will move 300 feet in a minute. We know only that at one particular instant it was moving at a rate which, if continued without change for a second, would have carried it through five feet in that time. This is not in any way inconsistent with its actual movement in the second being three feet or twenty feet instead of five, for its velocity, or rate of motion, may change altogether before the second is finished.

It has to be particularly remembered also that such a numerical value as that just given refers only to the *magnitude* of the velocity, and does not give any information about its *direction*, which, as we shall see, is in many cases equally important.

The velocity of a body is thus what may be called an "instantaneous" quantity. *At a certain instant* the body is moving at such and such a rate. The fact that this rate was quite different the instant before, and will be again quite different the instant after, does not in the least affect the matter. When, therefore, we represent the (instantaneous) velocity of a body by a line $A A_1$, we do not mean that the body actually moves from A to A_1 in a second, but only that when it is at A it is moving at a rate and in a direction which would bring it to A_1 in a second if only rate and direction continued unaltered for so long.

It happens, however, that we have often to concern ourselves with the *mean* velocity which a body has during a certain interval of time, that is to say, the mean of the

velocities which it has at each successive instant throughout that time, instead of its velocity at one instant only. But a velocity over any interval of time may be *uniform* or *varying*. In the former case the velocity is the same at every instant, and the mean velocity during the whole time is one and the same with the instantaneous velocity at any instant whatever during the time. If a body have a uniform velocity v_0 and pass through a distance s in t seconds the relation between the three quantities is simply

$$s = v_0 t.$$

This relation is equally true if v_0 be the *mean* velocity of a body which has had varying velocity during the time t ; when, that is, its velocities at different instants during the time have been different. But in that case the *actual* velocity might be v_0 at perhaps only one instant during the whole time, differing from it more or less at all other instants.

In the case of a varying velocity the rate of variation may itself (as we shall see in § 24) be uniform or varying. In the former case the mean velocity is the arithmetical mean between v_1 , the initial velocity, and v_2 , the final velocity,¹ or

$$v_0 = \frac{v_1 + v_2}{2} \qquad s = \frac{(v_1 + v_2)t}{2}$$

In such a case the mean velocity is therefore very easily found, and the actual velocity at any instant scarcely less easily. Fig. 92 (on p. 198) is a velocity diagram for such a case, where $v_1 = 1.5$ ft. per second and $v_2 = 5$ ft. per second. v_0 is therefore 3.25 feet per second, which is the actual

¹ If the sense of v_2 is opposite to that of v_1 it must have the minus sign prefixed to it, and $v_0 = \frac{v_1 + (-v_2)}{2} = \frac{v_1 - v_2}{2}$

velocity at the end of half the time interval, or two and a-half seconds from the start. v_0 is of course the mean height of the line whose ordinates represent the velocities.

But in the case of a body moving with some irregularly varying velocity, such as that shown by the diagram (Fig. 90) on p. 194, the mean velocity can only be found approximately by taking the mean of the actual velocities at a sufficient number of different points. The arithmetical mean of the initial and final velocities may, in such a case, differ to any extent from the real mean, and could not be substituted for it.

It is very important in what follows that the distinction which we have just enforced between the velocity of a body at a given instant and the mean of its velocities at a number of successive instants, should be kept in mind, a distinction which applies equally to angular and to linear velocities.

The linear velocity of a body, as a quantity having magnitude, sense, and direction, is a "directed quantity," or vector, which has been our justification for the representation of velocities by lines having just those properties, and which justifies us in applying all the graphic rules of vector addition, &c. to lines representing linear velocities.

We have now to look at the case where a body (plane motion being always presupposed) is *turning about a point at a finite distance*, so that its motion is a simple *rotation*. Here, as we have seen, the linear velocity of every point is proportional to its radius, so that all points not having the same radius have different linear velocities. But although the points of a rotating body have so different linear velocities, yet as long as the form of the body is not itself undergoing change, all points in it move through the same angle in the same time. Otherwise, as we said in § 7, different parts of it must have had different motions, and this is

impossible as long as the body remains rigid. The fact that all the points of a rotating body move through the same angle in the same time is expressed by saying that every point in it has the same angular velocity.

Just as either a foot, a yard, or a mile might be used as a unit for linear motion, so several different units might be used for angular motion—a revolution,¹ for instance, or a degree. There are practical conveniences, however, in taking for the unit of angular motion the angle subtended by an arc whose length is equal to its radius, which is $\left(\frac{360}{2\pi}\right)^\circ$ or about 57.3° . As the circumference of a circle of radius r is $2\pi r$, the number of units equal to r in one complete revolution is $\frac{2\pi r}{r} = 2\pi$, which is numerically equal to the distance moved through in one revolution by a point at unit radius. Further, if the body make n revolutions per second, it moves through $2\pi n$ angular units per second, which is again numerically equal to the number of feet traversed per second by a point at unit radius.

The number $2\pi n$ is called the angular velocity of the body, an expression which may be understood to mean either the rate at which the whole body is turning about its axis, expressed in angular units per second, or the rate at which any point in it having a radius equal to unity is moving, expressed in feet per second. The velocity in feet per second of any point in the body whose radius is r feet is obtained by multiplying the angular velocity of the body by the radius of the point, and is therefore equal to $2\pi n r$.

Where a body has a motion of translation only, it is often

¹ In cases where the revolution is used as the unit of angular motion the time unit is most commonly a minute instead of a second.

