SECTION I.

ON THE ACTUAL MOTION OF BODIES, AND ON THE FORCES CAPABLE OF PRODUCING ANY GIVEN MOTION.

Signification of the terms, Simple Rotatory Motion, and Angular Velocity.

THE only rotatory motion of which we have a clear idea, is that of a body which turns on an immoveable axis. For we see plainly all the circumferences of the circles which the different points of the body describe about this axis, and which they can really describe at the same time, without changing in any respect their relative position, or what we may denominate the form of the body.

We have an equally clear idea of the quantity or measure of this rotatory motion; for since all the points in it describe similar arcs in the same time, the ratio of the velocity of a point to the radius of the circle which it describes is the same for all points, and it is this constant ratio which forms the measure, or, as it is called, the *angular velocity* of rotation. (1)

(1) Let OPp (fig. 1) be the axis of rotation, R, r, any two points describing the circles RR', rr', which must lie in planes perpendicular to Op, and therefore parallel to each other.

Let RR', rr', be arcs described uniformly in the same time (t).

Draw PS parallel to pr, and when pr comes into the position pr', let PS' be the corresponding position of PS.

Then $\angle SPS' = \angle rpr'$. (Euc. XI. 10.)

But since the body is rigid,

$$SR = S'R',$$

and therefore, $\angle RPR' = \angle SPS' = \angle rpr'$.

Hence every point in the body describes round Op in the time (t) an angle $= RPR'$.

Let (ω) be the angle described in 1″ by every point,

then $\angle RPR' = t\omega$,

and the actual velocity (v) of $R = \dfrac{RR'}{t}$

$$= \frac{PR \,.\, \angle RPR'}{t} = PR \,.\, \omega,$$

and $\dfrac{v}{PR} = \omega$, which is the same for every point, is called the *Angular Velocity* of the body, or the *Velocity of Rotation*.

Composition of Rotatory Motion.

From these simple notions, and from the primary elements of Geometry, we may conclude that if, from the influence of any two separate causes, a body tended to turn at the same time round the two sides of a parallelogram, with two angular velocities respectively proportional to the lengths of these sides, the body would turn round the diagonal, with an angular velocity proportional to the length of this diagonal. (2)

(2) Let the straight lines Oa, Ob, (fig. 2.) lying in the plane of the paper intersect each other in O, and suppose two impulses to act simultaneously upon a body, one of which would cause it to turn about Oa with an angular velocity $= e \,.\, OA$, and the other to turn about Ob with an angular velocity $= e \,.\, OB$. Draw AP, BP, parallel to OB, OA, and PM, PN perpendicular to OA, OB; and let QPQ' be the circle which the point P would describe, if the first impulse were communicated singly, about Oa,

RPR' ....
........................second ....
.... Ob.

Then the planes in which these circles lie will both of them be perpendicular to the plane bOa passing through the axes, and therefore their intersection pPp' will be perpendicular to this plane, and therefore to each of the lines PM, PN, which are the radii of the circles, and will therefore be a tangent to both circles at P.

The first impulse therefore, would make P tend to move in the direction Pp with a velocity $= MP \times$ (angular velocity round Oa)

$$= e \,.\, OA \,.\, MP = e \,.\, OA \,.\, OB \sin BOA,$$

and the second in the direction Pp' with a velocity $= NP \times$ (angular velocity round Ob)

$$= e \,.\, OB \,.\, NP = e \,.\, OB \,.\, OA \sin BOA,$$

that is, in a direction exactly opposite, with the same velocity. If therefore these impulses were communicated together, the point P would remain at rest; and the same may evidently be proved of every other point in the line OP, which would therefore remain entirely at rest in consequence of the two impulses.

Draw AK, AL perpendicular to OB, OP. Then it is evident that the first impulse would not cause any tendency to motion in the point A, and the second would cause it to move in the direction of the tangent to a circle whose radius is AK, and whose plane is perpendicular to Ob, that is, in a direction perpendicular to the plane bOa, with a velocity $= e \,.\, OB \,.\, AK$. But in consequence of the two impulses, the line OP remains fixed in space, and since the body is rigid, the distance AL is invariable, and the direction of A's motion coincides with the tangent to a circle whose radius is AL, and whose plane is perpendicular to OP. The point A will therefore describe about L the circle SAS', with an angular velocity $= \dfrac{e \,.\, OB \,.\, AK}{AL} = e \,.\, OP$, (by similar triangles OPN, PAL.) And in the same manner, it may be shewn that every point in OA will describe a circle, in a plane perpendicular to OP, with an angular velocity $= e \,.\, OP$. Hence the whole line OA, and consequently the rigid body in which it lies, will turn about OP with an angular velocity $= e \,.\, OP$.

The same reasoning clearly holds if an interval occur between the communication of the impulses.

The velocities of the highest points Q' and R of the circles described by a point lying within the $\angle\, bOa$, are here supposed to be in the same direction, when resolved perpendicularly to a plane bisecting $\angle\, bOa$; and the rotatory motions are consequently said to be in the same direction.

If they are in opposite directions, we must take $OB' = OB$ (fig. 3.) on the other side of OA and complete the parallelogram, when OP' may be shewn as before to be the new axis, and $e \, . \, OP'$ the angular velocity of the body about it.

Whence it follows that the rotatory motions about different axes which intersect in any point are compounded in precisely the same manner as simple forces applied at the point.

And this similarity of composition is not confined to rotatory motions about axes which intersect; but what is very remarkable, it extends to rotatory motions about axes situated any how in space.

Thus rotatory motions about two parallel axes are compounded into a single one equal to their sum, about an axis parallel to them, which divides their distance in the inverse ratio of the component rotatory motions. (3)

(3) Let the axes aa', bb' (fig. 4.) be as before in the plane of the paper. Draw AB perpendicular to them, and let

$$\text{angular velocity round } aa' = e \, . \, BP,$$
$$\text{.......................} \quad bb' = e \, . \, AP,$$

then as before,

Velocity of P upwards, in a direction perpendicular to the plane of the paper, due to the rotatory motion round aa', $= AP \times$ (angular velocity round aa')

$$= e \, . \, BP \, . \, AP.$$

Velocity downwards, due to the rotatory motion round bb',

$$= e \, . \, AP \, . \, BP.$$

And P, and similarly every point in pp', remains at rest, and therefore pp' becomes the new axis of rotation.

And the velocity of A perpendicular to AP, which is that due to the rotatory motion round bb', $= e \,.\, AP \,.\, AB$; therefore velocity of A round P, or velocity of the body's rotation round pp',

$$= \frac{e \,.\, AP \,.\, AB}{AP} = e \,.\, AB = e \,.\, (BP + AP),$$

$$= \text{(velocity of rotation round } aa')$$
$$+ \text{(velocity of rotation round } bb').$$

What is meant by saying that the rotatory motions are in the same direction is evident from the last note: that is, that the motions of Q' and R are in the same direction.

If these are in opposite directions, the resultant rotatory motion is equal to their difference, and the position of the axes is determined by the same laws as that of the resultant of two parallel forces acting in opposite directions. (4)

(4) If the rotatory motions are in opposite directions, the motion of r', (fig. 5.) the highest point of the circle described by a point P' round bb', will be in an opposite direction to that of Q', which we may suppose to be the same as in the last note.

Suppose the angular velocities to be unequal, and let that round aa' be the greater. In BA produced take a point P, such that

angular vel. round aa', which we may call ω_a, $= e \,.\, PB$,

.... bb', ... ω_b, $= e \,.\, PA$,

then it is evident that the motion of R' will be parallel to that of r', and the motion of P round bb' will be in the direction RPR', while the motion round aa' is in the direction $Q'PQ$; and therefore as before,

velocity of P downwards $= \omega_a \,.\, PA = e \,.\, PB \,.\, PA$,

.... upwards $= \omega_b \,.\, PB = e \,.\, PA \,.\, PB$;

therefore pp' is the new axis of rotation, and the velocity of rotation round pp', or ω_p,

$$= \frac{e.PB.AB}{PB} = e.AB = ePB - ePA = \omega_a - \omega_b.$$

We may therefore conclude that the resultant of two rotatory motions ω_a, ω_b, about parallel axes aa', bb' is a rotatory motion about an axis parallel to them in the same plane, which $= \omega_a \pm \omega_b$, according as the component rotatory motions are in the same or opposite directions; the distance of the axis from bb' being equal to BP

$$= BA . \frac{e.BP}{eBP \pm eAP} = BA . \frac{\omega_a}{\omega_a \pm \omega_b},$$

BA being measured in a positive direction.

If the motions are in opposite directions and $\omega_b > \omega_a$, BP will be negative, and P will lie on the other side of B.

If $\omega_a = \omega_b$, BP will be infinite, and every point will describe a circle of infinite radius. This case is considered in the next note. (*See Pritchard's Couples, Appendix*, Prop. C.)

If these two parallel and opposite rotatory motions are equal, they can never be reduced to a single one. They form in that case what may be called a *Couple of Rotatory Motions:* a rotatory motion *sui generis*, and which can never be reduced to a simple rotatory motion about any axis whatever. And in fact it is easy to see that the result of such a couple would be to give the body a simple motion of translation in space, in a direction perpendicular to the plane of the couple, and measured by its *moment*, that is, by the product of one of