SECTION II.

SOLUTION OF THE PROBLEM OF THE ROTATION OF FREE BODIES.

It is evident that the axis of rotation which we denominate *the instantaneous axis* remains immoveable only for a single instant. For from the rotatory motion itself there arise, for each of the equal molecules of the body, centrifugal forces respectively proportional to, and in the direction of, the radii of the circles which these molecules tend to describe. Now the axis of which we speak being, by hypothesis, not a principal axis, these centrifugal forces will not balance each other. When removed parallel to themselves to the centre of gravity, they give it is true a resultant which equals nothing, but the resultant couple does not vanish. There arises therefore from the rotatory motion itself an *accelerating* couple, the action of which impresses on the body at each instant an infinitely small rotatory motion, which is compounded with the actual rotatory motion of the body, and causes both the magnitude and the axis of it to vary. (15)

(15) The *whole* effect of the couple being, as we saw in the last note, to produce a rotatory motion about GP, with an angular velocity $\omega_p = \frac{M\cos\theta}{P}$, we may consider the body in motion as acted on by no external force. And in such case the tension of the rod (QR),

connecting the molecule m at Q with the axis GP, which is for the instant immoveable,

$$= m \cdot \frac{(\text{velocity of } Q)^2}{QR}$$

$$= m \cdot QR \cdot (\omega_p)^2$$

applied at R in the direction RQ; which is equivalent to

$$\text{a force} = (\omega_p)^2 \cdot mQR \text{ at } G, \text{ perpendicular to } GP,$$

and a couple whose moment is $(\omega_p)^2 \cdot mQR \cdot GR$, and whose plane passes through GR.

And it is clear from the reasoning in note (11) that the resultant of all these centrifugal forces at G will vanish, by the property of the centre of gravity.

And the resultant of all the couples will be a couple, in a plane passing through the axis, equal to

$$\omega_p \times (\text{resultant of all the couples } \omega_p \cdot mQR \cdot GR)$$

$$= \omega_p \times \{\text{resultant of the couples (ii) for the axis } GP\}$$

which does not vanish; since the axis is not a principal one.

In order to investigate the motion of the body, it is necessary therefore to commence by finding this accelerating couple which arises from the centrifugal forces. Now it is very easy to *see directly*, or to conclude from the principle of the *conservation of couples**, that if we take two lines to represent respectively, the axis and magnitude of the impressed couple, and the axis and magnitude of the instantaneous rotatory motion, the accelerating couple due to the centrifugal forces is always represented in magnitude and position

* See Appendix.

by the surface of the parallelogram constructed on these two lines: a simple and remarkable theorem, which includes the whole theory of rotatory motion, and which when interpreted analytically gives immediately those three elegant equations which we owe to Euler, but which are ordinarily obtained by long and circuitous operations only. (16)

(16) It is clear that since each centrifugal force is proportional, and in a direction perpendicular, to the corresponding effective force which must have originally acted on the particle m to produce its motion, the resultant couple will be in a plane perpendicular to that of the resultant couple of the effective forces, and therefore, during the first instant, perpendicular to that of the impressed couple M.

The plane of the accelerating couple will therefore, during the first instant, pass through the axis of M as well as through the axis of rotation.

And if (l) be a linear unit, we may take GP (fig. 12.) $= l \,.\, \omega_p$,

and GM in the axis of the couple $= \dfrac{M}{l}$.

Then the area of the parallelogram Gm, which is manifestly in the plane of the accelerating couple,

$$= GP \,.\, GM \,.\, \sin PGM$$
$$= \omega_p \,.\, M \sin\theta \,;$$

which from notes (14) and (15) we know to be the magnitude of the couple, *during the first instant.*

And since the body is acted on by no external force, whatever be the position into which it is brought by the action of the centrifugal forces, we may assume that the general resultant of the *elementary effective* forces, which produce its motion round the instantaneous axis, will always be the same; that is, that it will always be a couple equivalent to, and in the same plane with M. If therefore θ' be the angle which the instantaneous axis makes with the axis of the couple at any time, $M \cos \theta'$ and $M \sin \theta'$ will still be the respective resultants of the couples (iii) and (ii) for that axis. Hence the theorem holds generally.

For the equations of Euler see Appendix.

The accelerating couple arising from the centrifugal forces being always situated therefore in the plane drawn through the axis of the impressed couple and the instantaneous axis, the straight line on which it tends to make the body turn is the diameter conjugate to the plane of these two axes, and consequently is the diameter which is at the same time conjugate to the instantaneous axis and its projection on the plane of the couple. Whence I conclude in the first place, that *the axis of the rotatory motion caused by the centrifugal forces always lies in the plane of the impressed couple.* (17)

(17) The instantaneous axis being conjugate to the plane of the impressed couple, is conjugate to every pair of conjugate diameters in the section of the ellipsoid made by that plane, and therefore to the pair, one of which is GU (fig. 13.) its *projection* on that plane; i. e. the intersection of the plane of the resultant couple of the centrifugal forces, which is perpendicular to the plane of the impressed couple, with that plane. Therefore the axis of instantaneous rotation and its projection on the plane of the impressed couple are conjugate axes of MPU the section of the ellipsoid made by the

plane of the centrifugal couple, and therefore the axis *GQ* about which this couple tends to make the body turn is conjugate to this pair. It must therefore be the diameter conjugate to *GU* in *UGN*, the section made by the plane of the impressed couple.

If then we take two lines, one of which represents this infinitely small motion, and the other the actual motion, and complete the parallelogram, in order to obtain, in the diagonal, the line which represents the rotatory motion at the end of an instant, it is clear that the extremity of this diagonal remains always at the same height above the plane of the couple. Whence we deduce immediately these two corollaries; the first, that

Throughout the whole course of the motion, the angular velocity is proportional to the length of the diameter passing through the pole of instantaneous rotation on the surface of the central ellipsoid: the second, that

The plane of the couple, considered always as a tangent at the pole, remains constantly at the same distance from the centre of this ellipsoid. (18)

(18) Let *GQ* (fig. 13.) be the axis about which the centrifugal forces tend to make the body turn, *PRQ* the section of the ellipsoid made by the plane containing the instantaneous axis *GP*, and *GQ*. Then by the last note *GP*, *GQ*, are conjugate axes of this ellipse; and therefore the tangent at *P*, and for a small distance the arc *PR*, is parallel to *GQ*.

Take the very small line $Gq = \frac{GP}{\omega_p}$ (vel. round *GQ*) and complete the parallelogram *GR*. Then *R* will be a point in the curve, and therefore the next pole of instantaneous rotation; and the height of it above GQ is manifestly the same as that of P.

$$\text{Also velocity round } GR = \omega_p \frac{GR}{GP},$$

we see moreover from the reasoning in note (16) and from the theorem in page 37, that the body will come into such a position that the tangent plane at the pole R shall be parallel to the plane of the couple originally impressed: which in the original position of the body we may suppose to be a tangent at P.

But the centre is fixed in absolute space, and the plane of the couple always parallel to itself; therefore *this plane, which is always a tangent at the instantaneous pole of rotation, is an invariable plane, fixed in absolute space.*

Therefore the motion of the body, or what is the same thing the motion of the central ellipsoid, is such, that this ellipsoid remains in contact with a plane fixed in absolute space: that is, it turns at every instant about the radius vector at the point of contact, with an angular velocity proportional to the length of this radius.

This ellipsoid therefore rolls without sliding on the fixed plane abovementioned; for since all its motion consists in turning for an instant on the line drawn from the centre to the point of contact, the ellipsoid brings at the end of this instant a new point into contact with the plane; and this new point, which becomes the pole of rotation for the instant following, remains in its turn immoveable for this instant, and so on to infinity; whence it is manifest that none of the