SECTION III.

DEVELOPEMENT OF THE SOLUTION.

The two Curves described by the Instantaneous Pole of Rotation.

THE above illustration of the rotatory motion of a body leads us at once, and as it were by the hand, to the calculations necessary to measure all the different affections of this motion.

And first this succession of points, at which the central ellipsoid comes into contact with the fixed plane of the impressed couple, traces on the surface of the ellipsoid the path of the instantaneous pole in the interior of the body, and the corresponding succession of points on the fixed plane traces its path in absolute space. We can therefore determine immediately these two curved lines, and consider them as the bases of two conical surfaces having the same vertex, one of which, moving with the body, would by rolling on the other, which is fixed in absolute space, cause in the body the precise motion with which it is endued.

To find the first curve we have only to determine the succession of points in which the ellipsoid is touched by a plane which is always at the same distance from its centre; or what is the same thing, which touches a concentric sphere whose radius is equal to the given distance.

While this plane traces on the ellipsoid the path of the instantaneous pole, we may remark that it traces on the sphere the path that the pole of the couple, which is fixed in space, would appear to describe in the interior of the moveable body; a curve of the same nature which we shall have also occasion to consider.

But to speak of the first only: we see that it is *a re-entering curve of double curvature*, having like the ellipse *four principal vertices*, at which it is divided into four equal and symmetrical parts; a species of elliptical *wheel*, whose *axle* is always either the *greatest* or *least radius* of the central ellipsoid, according as the radius of the sphere is given greater or less than the mean radius of the ellipsoid. This curve of double curvature is projected in *a complete ellipse* on the plane perpendicular to the axis which forms its *axle*, in *an elliptic arc* on the other plane, and always in *an hyperbolic arc* on the plane perpendicular to *the mean radius*. (19)

(19) Let Pg (fig. 14.) be a perpendicular section of the tangent plane touching the ellipsoid in P and a concentric sphere whose radius $Gg = r$ in g.

Let x, y, z, be the co-ordinates of P,
x', y', z', ... of g,
x'', y'', z'', ... of any point in the tangent plane.

Then we have two equations to this plane; viz:

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} + \frac{zz''}{c^2} = 1,$$

$$\text{and } \frac{x'x''}{r^2} + \frac{y'y''}{r^2} + \frac{z'z''}{r^2} = 1,$$

which must coincide; therefore $\frac{x'}{r^2} = \frac{x}{a^2}$,

$$\text{and } \frac{x'^2}{r^4} = \frac{x^2}{a^4},$$

$$\frac{y'^2}{r^4} = \frac{y^2}{b^4},$$

$$\frac{z'^2}{r^4} = \frac{z^2}{c^4};$$

$$\text{Therefore } \frac{1}{r^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4};$$

$$\text{Also } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

which are the equations to the curve of double curvature. Let a be the greatest and c the least semi-axis. Then if $r > b$, the curve can never meet the plane perpendicular to the major axis;

$$\text{for if } x = 0, \quad z^2 = c^2 . \frac{\frac{1}{r^2} - \frac{1}{b^2}}{\frac{1}{c^2} - \frac{1}{b^2}},$$

a negative quantity. The curve therefore lies in this case wholly about the major axis. And *vice versâ* if $r < b$. Also the equation to the projection on the principal plane perpendicular to c is

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + c^2\left(\frac{1}{r^2} - \frac{x^2}{a^4} - \frac{y^2}{b^4}\right),$$

$$\text{or} \quad \frac{x^2}{a^2} \cdot \frac{1 - \frac{c^2}{a^2}}{1 - \frac{c^2}{r^2}} + \frac{y^2}{b^2} \cdot \frac{1 - \frac{c^2}{b^2}}{1 - \frac{c^2}{r^2}} = 1,$$

which is evidently the equation to an ellipse whose semi-axes are

$$a\sqrt{\frac{1 - \frac{c^2}{r^2}}{1 - \frac{c^2}{a^2}}}, \quad \text{and} \quad b\sqrt{\frac{1 - \frac{c^2}{r^2}}{1 - \frac{c^2}{b^2}}},$$

the first of which is always $< a$; and if $r < b$, the second is $< b$, or the projection is a complete ellipse, concentric to the principal section of the ellipsoid; but if $r > b$ the minor semi-axis is $> b$, and the projection is only part of an ellipse.

The equation to the projection perpendicular to the mean semi-axis is

$$\frac{x^2}{a^2} \cdot \frac{1 - \frac{b^2}{a^2}}{1 - \frac{b^2}{r^2}} + \frac{z^2}{c^2} \cdot \frac{1 - \frac{b^2}{c^2}}{1 - \frac{b^2}{r^2}} = 1,$$

which is manifestly the equation to an hyperbola, the coefficients of x^2 and z^2 having necessarily different signs.

The four vertices of this curve are the points where the radius vector, and consequently the velocity of rotation, attain their *maximum* and *minimum* values; and we may remark that the *maximum* always occurs when the instantaneous pole passes through the two vertices which lie in the *mean* principal plane of the ellipsoid, and the *minimum* when it passes through the other two vertices. (20)

(20) Since $GP = \sqrt{x^2 + y^2 + z^2}$,

$$0 = d_x GP = GP \,.\, d_x GP$$

$$= x + y d_x y + z d_x z,$$

also $$0 = \frac{x}{a^2} + \frac{y}{b^2} d_x y + \frac{z}{b^2} d_x z,$$

$$0 = \frac{x}{a^4} + \frac{y}{b^4} d_x y + \frac{z}{b^4} d_x z,$$

whence we derive separately $y = 0$, $x = 0$, which give possible values for GP when $r > b$, the former satisfying the conditions for a maximum, and the latter for a minimum; the value of the maximum radius (P) being

$$\sqrt{x^2 + z^2} = \frac{a^2 \sqrt{r^2 - c^2} + cr\sqrt{a^2 - r^2}}{r\sqrt{a^2 - c^2}},$$

and of the minimum (ρ),

$$\sqrt{x^2 + y^2} = \frac{a^2 \sqrt{r^2 - b^2} + br\sqrt{a^2 - r^2}}{r\sqrt{a^2 - b^2}}.$$

The second curve, being traced by that which rolls about the centre on the fixed plane of the couple, is therefore a plane curve which encircles the projection of the centre, forming equal and regular undulations corresponding to the equal and symmetrical arcs of the rolling orbit which produces it: it is a species of circular curve whose radius varies periodically, and which winds for ever between two concentric circles whose circumferences it touches alternately. (21)

(21) The projection (gp) of the radius vector GP on the fixed plane of the couple will evidently arrive at its maximum and minimum values cotemporaneously with GP; for when GP is greatest its inclination to this plane is least.

With centre g, (fig. 15.) the projection of G, and radii ρ and P, describe two circles; then the curve being perpendicular to the radius vector gp at the points where it attains its maximum and minimum values, will manifestly touch these circles at those points; that is, will touch them alternately, since the curve passes alternately the mean and major principal planes.

The consequences deduced in this and the last note for $r > b$ are easily adapted to the case when $r < b$.

If the angle at the centre which corresponds to two consecutive vertices of these equidistant undulations is commensurable with four right angles, the curve re-enters itself after a certain number of revolutions; and the instantaneous pole which describes it returns at once to the same position, both in the body and in space. But in the contrary case the curve never re-enters itself, and the pole, which always returns periodically to the same place in the body, can never return at the same time to the same point in space.

Such are the two curves described by the instantaneous pole, the one in the interior of the body, and the other in absolute space. And although these curves are of such different forms, yet since it is one and the same point which describes them both, the equations to them, between the radius vector and the arc, exactly coincide. (22)

(22) To find this equation to the two curves, we have

$$p^2 = x^2 + y^2 + z^2,$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

$$\frac{1}{r^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}.$$

Substituting in the second and third the values of z^2 derived from the first, multiplying the former by $\frac{1}{b^2} + \frac{1}{c^2}$, and subtracting, we have

$$x^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + \frac{p^2}{b^2 c^2} = \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{r^2};$$

$$\therefore\ x^2 = a^4 . \frac{\frac{b^2 c^2}{r^2} - \left(\frac{1}{b^2} + \frac{1}{c^2}\right) + p^2}{(a^2 - b^2)(a^2 - c^2)},$$

or $x = a' . \sqrt{\alpha^2 + p^2}$, suppose;

$$\therefore\ d_p x = \frac{a' p}{\sqrt{a^2 + p^2}}.$$

Similarly, $d_p y = \dfrac{b' p}{\sqrt{\beta^2 + p^2}}, \quad d_p z = \dfrac{c' p}{\sqrt{\gamma^2 + p^2}};$

$$\therefore\ d_p s, \text{ which} = \sqrt{(d_p x)^2 + (d_p y)^2 + (d_p z)^2}$$

$$= p\sqrt{\frac{a'^2}{a^2 + p^2} + \frac{b'^2}{\beta^2 + p^2} + \frac{c'^2}{\gamma^2 + p^2}}.$$

The *rolling cone* of which the first curve forms the base is simply *a right cone of the second degree;* but the *fixed cone* on which it rolls is a *transcendent cone*, whose surface undulates for ever about the fixed axis of the couple: it is a species of right circular cone, whose surface however is *fluted* according to the regular undulations of the curve which forms its base.

Proposed Names for the two Curves.

We know that a heavy body projected any how in space turns on its centre of gravity, exactly as if it were free from the action of gravity. The two remarkable curves therefore above described are presented constantly to our notice in the motion of projectiles, and merit names as much as the path of the centre of gravity which is called a parabola.

I propose therefore to give them the names of relative and absolute ***Poloids;*** or rather, in order to distinguish them by their respective forms,