APPENDIX.

I. *The Axis of Instantaneous Rotation.*

When a rigid body is in motion, it is turning, during every separate instant, about some straight line or other. (v. page 17.)

For an analytical proof of the existence of this line see Whewell's *Dynamics*. (Art. 120.)

The following is extracted, by permission of the Author, from Earnshaw's *Statics*. (Art. 109.)

"Let P, Q, (fig. 18.) be any two particles of a rigid body; PP', QQ', the paths which they describe during the same instant; A, B, the centres of curvatures of these paths; then the line joining A, B, will be the axis about which the whole body turns during this instant."

"For the lines which join the successive contempo raneous positions of P and Q, while they are respectively passing to P' and Q', will form a species of conical surface; and since, by reason of the rigidity of the body, they are all of the same length, the planes PAP', QBQ', in which the curves (PP', QQ') formed by their extremities lie, must be parallel. Now since P describes round A the angle PAP' in the same time that Q describes round B the angle QBQ', the trapezium $PABQ$ turns in the same time round AB and comes into the position $P'ABQ'$ (for $AP = AP'$, and $BQ = BQ'$, since A, B, are the centres of curvature of PP', QQ'). Consequently the motion of every particle of the body situated in PQ takes place about the axis AB."

"Hence the motions of the points P and Q take place about AB, and therefore AB must be perpendicular to PAP', QBQ', the planes of these motions. In like manner if the motion of any other particle R take place about a point C, AC must be perpendicular to the planes of motion PAP', RCR'; hence both AB and AC are perpendicular to PAP', which is impossible, (Euc. XI. 13.) unless they coincide; in which case C is a point in AB, and the motion of R takes place about AB; and since R is any particle, the motion of every particle takes place about AB, that is, the whole body turns during the instant about the straight line AB."

If any point in the body be fixed the axis must pass through this point.

For let O be the point and join AO; then we may consider PAO as a crooked but rigid rod moveable about the fixed point A, and it is manifest that while one extremity P moves through PP', the other cannot remain at rest unless it lies in AB.

In this case the motion of any one point P determines the motion of every other point.

For if PO be joined, the rod PO considered as a rigid body must be turning during every separate instant about some axis passing through O; and it is shewn in the course of the above demonstration that the plane of the motion of any other point Q in the rigid body is parallel to the plane of the motion of any point in OP. Therefore the motion of Q is about the same axis as that of OP; and it is clear from note (1) that the angular velocities are the same.

If the body is perfectly free, it has during every instant a simple rotatory motion about some axis passing through the centre of gravity; except in the case when all the particles move in equal and parallel straight lines, that is, when the body has a mere motion of translation.

To prove this it is necessary to establish the following Dynamical property of the centre of gravity.

If a motion of translation be communicated to a body which has a simple rotatory motion about any axis passing through its centre of gravity, the motions will subsist together, and each will continue to affect the body precisely as it would have done if the other had never existed.

Suppose a velocity v in the direction of a line which makes angles α, β, γ, with the co-ordinate axes to be communicated to every particle of the rigid system in note (11).

Then the resolved parts of the effective forces which act on a particle M situated at the point P are

$$-m\omega y + mv\cos\alpha, \text{ in the direction } Gx,$$
$$m(\omega x + v\cos\beta) \;\ldots\ldots\ldots\ldots\ldots\ldots\; Gy,$$
$$mv\cos\gamma \;\ldots\ldots\ldots\ldots\ldots\ldots\; Gz.$$

And the resolved parts of all the elementary effective forces are reducible to

(i.) Three forces applied at G; viz.

$$-\omega\Sigma(my) + v\cos\alpha\,\Sigma(m),$$

which by the property of the centre of gravity (if $\mu = \Sigma(m)$ the mass of the system)

$$= \mu v\cos\alpha, \text{ in the direction } Gx,$$
$$\omega\Sigma(mx) + v\cos\beta\,\Sigma(m) = \mu v\cos\beta \;\ldots\ldots\ldots\ldots\ldots\ldots\; Gy.$$
$$v\cos\gamma\,\Sigma(m) = \mu v\cos\gamma \;\ldots\ldots\ldots\ldots\ldots\ldots\; Gz.$$

(ii.) Two couples; viz.

$$\Sigma m z(-\omega y + v\cos\alpha) - \Sigma m x v \cos\gamma$$
$$= -\omega\Sigma m y z \text{ in the plane } zx,$$
$$\Sigma m z(\omega x + v\cos\beta) - \Sigma m y v\cos\gamma$$
$$= \omega\Sigma m x z \ldots\ldots\ldots\ldots\ldots\; zy.$$

(iii.) Two couples; viz:

$$\Sigma m x(\omega x + v\cos\beta) \text{ and } \Sigma m y(\omega y - v\cos\alpha),$$

which together $= \omega\Sigma m(x^2 + y^2)$ in the plane xy.

The whole elementary effective forces are therefore equivalent to a force equal to the resultant of the forces (i) $= \sqrt{\mu^2 v^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)} = \mu v$ applied at the centre of gravity, in a direction making angles α, β, γ, with the axes, and to the resultant of the two sets of couples (ii) and (iii); and it is manifest that if the rotatory motion were suppressed, the resultant force would still be μv, and the resultant couple would vanish; and therefore conversely that the consequence of applying a force $= \mu v$ at the centre of gravity of the body is to give every particle of it a mere motion of translation with a velocity v, in the direction of the force, whether the body has or has not a rotatory motion; and that on the other hand the couples (ii) and (iii), which are necessary and sufficient to produce a simple rotatory motion about an axis through G, remain the same, whatever be the value of v, and therefore of μv.

Suppose now that AB (fig. 19.) the line containing the centres of curvature of the paths of every particle of the body *in absolute space*, does not pass through the centre of gravity G of the body, and that the angular velocity of the body about it is ω. Then if about Gg a line through G parallel to AB, we suppose two angular velocities each $= \omega$ to be communicated to the body in opposite directions, we shall have a simple rotatory motion with an angular velocity ω about Gg and a motion of translation perpendicular to the plane $GgAB$; and since Gg passes through the centre of gravity these motions are independent, that is, the body has during the instant, a *simple* rotatory motion about an axis through G.

Hence also it is impossible for a free rigid body to have a simple rotatory motion about any axis which does not pass through the centre of gravity.

For the rotatory motion which it has for an instant about any such axis can always be resolved into a simple one about an axis through the centre and a motion of translation. Hence the axis of the screw spoken of in pp. 24, 25, always passes through the centre of gravity of the body.

We may also illustrate the above principles by a reference to the motions of the Earth and Moon in absolute space.

If the Earth had no rotatory motion about its own axis we might naturally consider SK (fig. 20.) perpendicular to the plane of the ecliptic as the instantaneous axis (AB figs. 18 and 19.). The motion about it would be equivalent at every instant to a motion of translation in the direction of a tangent to the Earth's orbit, and a simple rotatory motion about an axis through the centre of gravity of the Earth parallel to SK. In the course of a year the Earth would turn once round on this axis in the direction WLE, and to a spectator on the Earth's surface the Sun would appear to describe in the *same* direction a circle round the centre of gravity of the Earth in the plane of the ecliptic, and the fixed stars parallel circles in the *opposite* direction. And this motion would be totally independent of the motion of translation, of which the spectator would only be aware from the consideration that if it were suppressed the Sun's apparent motion would be in the *opposite* direction.

But no such apparent annual motion of the fixed stars is observed. We may therefore conclude that the Earth has no such simple rotatory motion, and that its motion round the Sun is a simple motion of translation of the centre of gravity; which would also appear from Dynamical considerations. It is observed that the Moon presents always the same face towards the Earth. Hence all the particles in it describe similar curves about a line through the centre of gravity of the Earth perpendicular to the Moon's orbit. This axis is therefore always an instantaneous axis of the Moon. Hence the Moon has a simple rotatory motion in the same direction with its motion of translation about an axis through its centre of gravity parallel to the instantaneous axis, and turns once round on this axis during a revolution of its centre of gravity round that of the Earth.

If now we suppose one or more rotatory motions to be communicated about other axes through the centre of gravity, the motion of translation at every instant will not be affected, and the simple rotatory motion about the centre of gravity at every instant, upon which the appearances of the heavens depend, will be compounded of these.

II. *Principal Axes and Moments of Inertia.*

Through every point in a material system at least three straight lines may be drawn, in directions mutually at right angles, for which, when they are severally taken as the axis of z, each of the quantities $\Sigma(mxz)$, $\Sigma(myz)$ vanishes.

In a rigid body or system these lines are called principal axes. (v. page 28.)

If GP (fig. 11.) make angles α, β, γ, with the co-ordinate axes, and the co-ordinates of Q, the position of any particle m, be x, y, z; as in note (13).

$$\text{Moment round } GP\ (P) = \Sigma m\,(QR)^2$$

$$= \Sigma m\,(x^2 + y^2 + z^2 - GR^2)$$

$$= \Sigma m x^2 + \Sigma m y^2 + \Sigma m z^2$$

$$- \Sigma m x^2 \cos^2 \alpha - \Sigma m y^2 \cos^2 \beta - \Sigma m z^2 \cos^2 \gamma$$

$$- 2\Sigma m x y \cos \alpha \cos \beta - 2\Sigma m x z \cos \alpha \cos \gamma$$

$$- 2\Sigma m y z \cos \beta \cos \gamma.$$

Let $A = \Sigma m y^2 + \Sigma m z^2$, which is manifestly the moment round Gx,

$A' = \Sigma m y z$, and so on for the other axes.

$$\text{Then} \quad P = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma$$

$$- A' \cos \beta \cos \gamma - B' \cos \alpha \cos \gamma - C' \cos \alpha \cos \beta.$$

And if we take any point x, y, z in GP at a distance p from G, we shall have

$$Pp^2 = Ax^2 + By^2 + Cz^2 - A'yz - B'xz - C'xy.$$

Now the value of P evidently depends upon the values of α, β, γ, that is upon the position of GP; and p being arbitrary we may take it equal to any function of the same quantities;

$$\text{let it} = \frac{1}{\sqrt{n \,.\, P}};$$

$$\therefore\ Pp^2 = \frac{1}{n},$$